

DETERMINANTS ASSOCIATED TO ZETA MATRICES OF POSETS

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ABSTRACT. We consider the matrix $\mathfrak{Z}_P = Z_P + Z_P^t$, where the entries of Z_P are the values of the zeta function of the finite poset P . We give a combinatorial interpretation of the determinant of \mathfrak{Z}_P and establish a recursive formula for this determinant in the case in which P is a boolean algebra.

§1. INTRODUCTION

The theory of partially ordered sets (posets) plays an important role in enumerative combinatorics and the Möbius inversion formula for posets generalizes several fundamental theorems including the number-theoretic Möbius inversion theorem. For a detailed review of posets and Möbius inversion we refer the reader to [S1], chapter 3, and [Sa]. Below we provide a short exposition of the basic facts on the subject following [S1].

A partially ordered set (*poset*) P is a set which, by abuse of notation, we also call P together with a binary relation, called a *partial order* and denoted \leq , satisfying:

- (1) $x \leq x$ for all $x \in P$ (reflexivity).
- (2) If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).
- (3) If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

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Two elements x and y are *comparable* if $x \leq y$ or $y \leq x$. Otherwise they are *incomparable*. We write $x < y$ to mean $x \leq y$ and $x \neq y$.

Examples.

- (1) Let $n \in \mathbb{N}$. The set $[n] = \{1, 2, \dots, n\}$ with the usual order forms a poset in which any two elements are comparable. Such a poset is called a *chain* or a totally ordered set.
- (2) Let $n \in \mathbb{N}$. Consider the poset P_n of subsets of $[n]$ under the inclusion relation. This poset is called a *boolean algebra* of rank n . In [S1] it is denoted by $2^{[n]}$.
- (3) Let $n \in \mathbb{N}$. The set D_n of all positive divisors of n forms a poset under the order defined by $i \leq j$ in D_n if j is divisible by i . This poset is called the *divisor poset*.

A closed *interval* $[x, y]$ is defined whenever $x \leq y$ by $[x, y] = \{z \in P : x \leq z \leq y\}$. The empty set is not regarded as an interval. Denote by $Int(P)$ the set of intervals of P . If $f : Int(P) \rightarrow \mathbb{C}$, we write $f(x, y)$ for $f([x, y])$.

The *incidence algebra* $I(P)$ of P is the \mathbb{C} -algebra of all functions $f : Int(P) \rightarrow \mathbb{C}$ under the convolution given by

$$fg(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y).$$

It is known that $I(P)$ is an associative \mathbb{C} -algebra with identity

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

The *zeta function* $\zeta \in I(P)$ of a poset P is defined by $\zeta(x, y) = 1$ for all $x \leq y$ in P . If P is a locally finite poset (i.e. every interval in P is finite), the zeta function ζ is invertible in the algebra $I(P)$. Its inverse is called the *Möbius function* of P and is denoted by μ . Note that one can define μ inductively by

$$\mu(x, x) = 1, \text{ for all } x \in P,$$

$$\mu(x, y) = - \sum_{x \leq z < y} \mu(x, z), \text{ for all } x < y \text{ in } P.$$

Proposition 1. (*Möbius Inversion Formula*) Let P be a finite poset and $f, g : P \rightarrow \mathbb{C}$. Then

$$g(x) = \sum_{y \leq x} f(y) \text{ for all } x \in P \Leftrightarrow f(x) = \sum_{y \leq x} g(y) \mu(y, x) \text{ for all } x \in P.$$

PROOF. See [S], pg. 116. \square

There is also a dual form of the Möbius Inversion Formula. Moreover, the formula works for more general posets. All that is needed is that every principal order ideal be finite.

Remark. If P is the divisor poset of Example (3), the proposition above becomes the number-theoretic Möbius Inversion Theorem. If P is the boolean algebra of Example (2), the proposition gives the Principle of Inclusion-Exclusion. Finally, if P is the chain of Example (1), the proposition becomes the Fundamental Theorem of the Difference Calculus. For the aforementioned theorems see [Sa].

For the remainder of the article, P will be a poset with n elements and the partial order denoted by \leq . We choose a labelling x_1, x_2, \dots, x_n of the elements of P such that $x_i < x_j \implies i < j$.

Definition. The *zeta matrix* Z_P of a poset P is defined as the $n \times n$ matrix with entries

$$(Z_P)_{ij} = \begin{cases} 1 & \text{if } x_i \leq x_j \\ 0 & \text{otherwise} \end{cases}$$

Observe that, with the chosen labelling, the zeta matrix is unipotent upper triangular. Its non-zero entries are the values of the zeta function.

We define the matrix \mathfrak{Z}_P by $\mathfrak{Z}_P = Z_P + Z_P^t$. In Section §2 we give a combinatorial interpretation of the determinant of \mathfrak{Z}_P . Section §3 refines the interpretation in the case in which P is a chain. The value of the determinant in this case is $n + 1$. The main theorem of the paper evaluates the determinant of $\mathfrak{Z}_n := \mathfrak{Z}_{P_n}$ when P_n is the boolean algebra of rank n . More specifically, in Section §4, we prove the following recursive formula on n .

Main Theorem. *If $n \geq 3$ is odd, then $\det(\mathfrak{Z}_n) = 0$. If n is even, then $\det(\mathfrak{Z}_n) = 2^{\alpha_n}$ where $\alpha_2 = 2$, and $\alpha_n = 4\alpha_{n-2} - 2$ for $n \geq 4$ and even.*

Consider also the matrix \mathfrak{M}_P defined by $\mathfrak{M}_P = M_P + M_P^t$, where $M_P = Z_P^{-1}$. The non-zero entries of M_P are the values of the Möbius function. We refer to M_P as the *Möbius matrix* of the poset P . We have the following theorem.

Theorem 1. $\det(\mathfrak{M}_P) = \det(\mathfrak{Z}_P)$.

PROOF. The theorem is a direct consequence of the following lemma. \square

Lemma 1. *Let U be an $n \times n$ matrix such that $\det(U) = 1$ and let $V = U^{-1}$. Then, $\det(U + U^t) = \det(V + V^t)$.*

PROOF. We have

$$V^t + V = (U^{-1})^t + U^{-1} = (U^{-1})^t U U^{-1} + (U^t)^{-1} U^t U^{-1} = (U^{-1})^t [U + U^t] U^{-1}.$$

Thus $\det(V + V^t) = \det((U^{-1})^t) \det(U + U^t) \det(U^{-1})$ and since $\det(U) = 1$, we have $\det(V + V^t) = \det(U + U^t)$. \square

§2. COMBINATORIAL INTERPRETATIONS OF $\det(\mathfrak{Z}_P)$

In this section, we will give two related combinatorial interpretations of $\det(\mathfrak{Z}_P)$. The first involves collections of disjoint cycles in a directed graph D_P associated to P . The second involves cycle decompositions of elements of a certain subgroup S_n^P of the symmetric group S_n associated to P .

Consider a poset P as in the previous section with $|P| = n$. The matrix $Y_P = Z_P - I_n$, in which the diagonal entries of Z_P are replaced by 0, can be interpreted as the adjacency matrix of a *directed graph* (digraph) G_P associated to the strict order relation $x < y$ in P . The vertices of G_P are the elements of P , and there is a directed edge from x to y if and only if $x < y$. Similarly, the matrix $\mathfrak{Y}_P = Y_P + Y_P^t$ is the adjacency matrix of the directed graph D_P in which there are edges in *both* directions between each pair of distinct comparable elements $x, y \in P$. Then we have

$$\mathfrak{Z}_P = Z_P + Z_P^t = Y_P + Y_P^t + 2I_n = \mathfrak{Y}_P + 2I_n,$$

and $\det(\mathfrak{Z}_P) = \det(\mathfrak{Y}_P + 2I_n) = \chi(-2)$ where $\chi(t)$ is the characteristic polynomial $\chi(t) = \det(\mathfrak{Y}_P - tI_n)$ of the matrix \mathfrak{Y}_P . As in [M], we will also call this the characteristic polynomial of the graph D_P .

If $P = \{x_1, \dots, x_n\}$ and $\mathfrak{Y}_P - tI_n = (m_{ij})$, then by the definition of the determinant

$$\chi(t) = \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} \prod_{j=1}^n m_{j\sigma(j)}.$$

Then we have

$$\prod_{j=1}^n m_{j\sigma(j)} = \begin{cases} 0 & \text{if } x_j \text{ not comparable to } x_{\sigma(j)} \text{ for some } j, \\ (-t)^{k_\sigma} & \text{otherwise,} \end{cases}$$

where k_σ is the number of fixed points of the permutation σ .

Let $S_n^P = \{\sigma \in S_n : x_j \text{ comparable in } P \text{ to } x_{\sigma(j)}, \text{ for all } j = 1, \dots, n\}$. Then combining the last two displayed formulas

$$(1) \quad \chi(t) = \sum_{\sigma \in S_n^P} (-1)^{\text{sign}(\sigma)} (-t)^{k_\sigma}.$$

As shown in the work of Harary and Mowshowitz (see [H] and [M]), there is an interesting relation between the coefficients of $\chi(t)$ and cycles in the graph D_P . Namely, if we write

$$\chi(t) = \sum_{i=0}^n a_i (-t)^{n-i},$$

then $a_0 = 1$ (the coefficient of $(-t)^n$ equals 1 since the identity permutation in S_n is the only permutation with exactly n fixed points) and Theorem 1 of [M] demonstrates that for $i \geq 1$,

$$(2) \quad a_i = \sum \left(\prod_{j=1}^r (-1)^{i_j+1} \right) f_{D_P}(i_1, \dots, i_r),$$

where the summation is taken over all partitions $i = i_1 + \dots + i_r$ with $i_j > 0$ for all j , and $f_{D_P}(i_1, \dots, i_r)$ is the number of collections of disjoint directed cycles in D_P of lengths i_1, \dots, i_r . Thus

$$a_i = \sum (-1)^{i+r} f_{D_P}(i_1, \dots, i_r).$$

Note that in fact $i_j \geq 2$ for all j since D_P contains no loops (no edges from a vertex to itself). Therefore $a_1 = 0$, which corresponds to the fact that there are no permutations in S_n with exactly $n - 1$ fixed points.

Example. To illustrate the relation between (1) and (2), let $P = P_2$ be the boolean algebra of rank 2. The ordering of the elements of P is given by the correspondence

$$\emptyset \longleftrightarrow 00, \quad \{1\} \longleftrightarrow 01, \quad \{2\} \longleftrightarrow 10, \quad \{1, 2\} \longleftrightarrow 11.$$

Thus we have the labelling

$$P = \{x_1 = 00, x_2 = 01, x_3 = 10, x_4 = 11\}.$$

Then

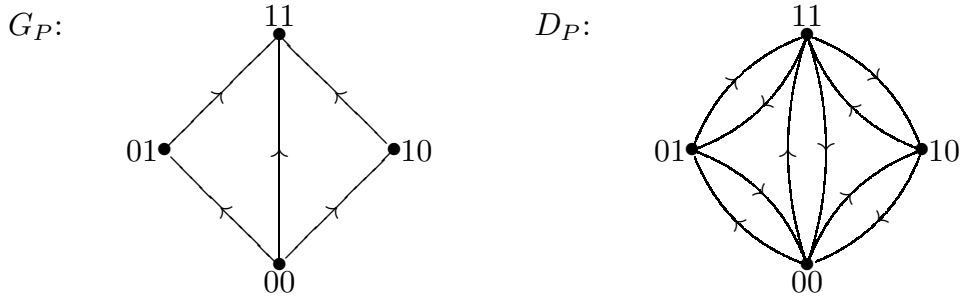
$$\mathfrak{Z}_P = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

and

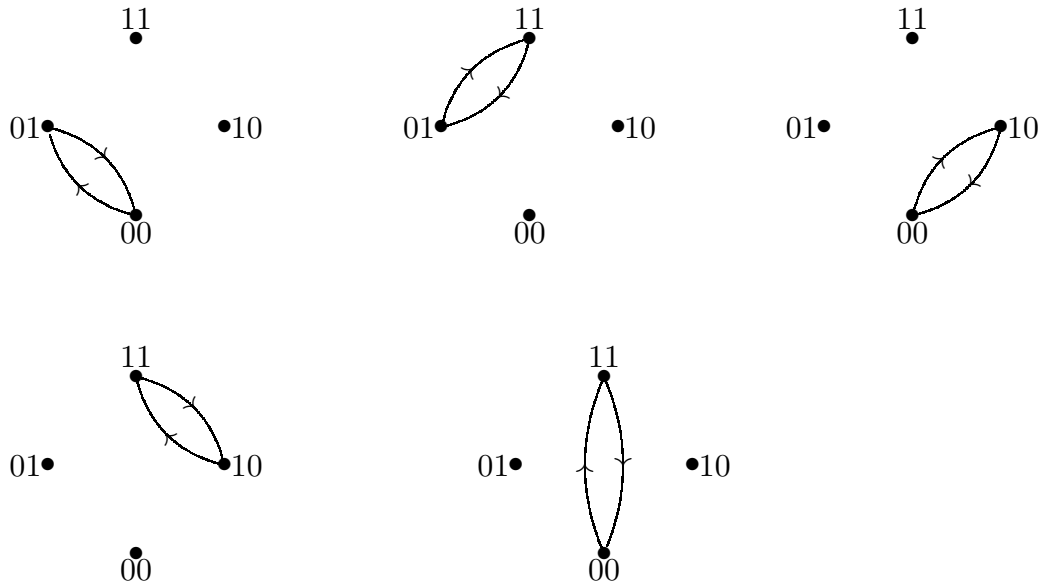
$$\begin{aligned}\chi(t) &= \det(\mathfrak{Y}_P - tI_4) = \det \begin{pmatrix} -t & 1 & 1 & 1 \\ 1 & -t & 0 & 1 \\ 1 & 0 & -t & 1 \\ 1 & 1 & 1 & -t \end{pmatrix} \\ &= \sum_{i=0}^4 a_i (-t)^{4-i} = (-t)^4 - 5(-t)^2 + 4(-t),\end{aligned}$$

where the a_i 's are defined as above, so $a_0 = 1$, $a_1 = 0$, $a_2 = -5$, $a_3 = 4$, $a_4 = 0$.

The digraphs G_P and D_P are given below.

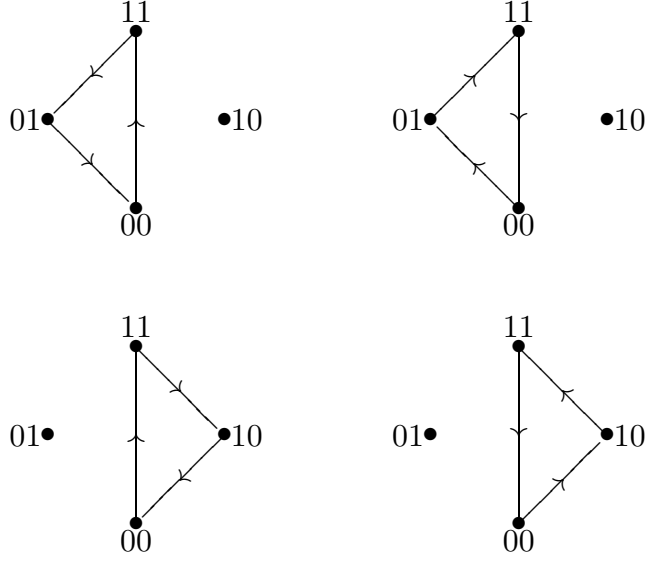


We obtain one term of $(-t)^2$ for each permutation in S_4^P with exactly 2 fixed points. Since each part of the partition must be at least 2, the only allowable partition of 2 is $2 = 2$ and $f_{D_P}(2)$ counts the following collections of cycles in D_P which correspond to the transpositions in S_4^P :



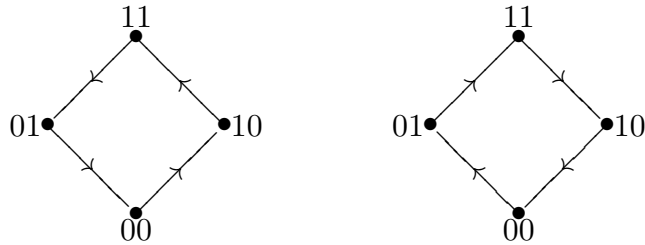
This corresponds to the coefficient $a_2 = -5$.

We obtain one term of $(-t)^1$ for each permutation in S_4^P with exactly 1 fixed point. The only allowable partition of 3 is $3 = 3$ and $f_{D_P}(3)$ counts the following collections of cycles in D_P which correspond to the 3-cycles in S_4^P :

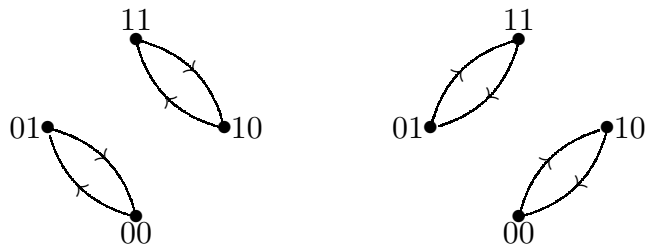


This corresponds to the coefficient $a_3 = 4$.

Finally, we obtain one term of $(-t)^0 = 1$ for each permutation in S_4^P with no fixed points. Such permutations are either 4-cycles or products of 2 transpositions. The only allowable partitions are $4 = 4$ and $4 = 2 + 2$. Then $f_{D_P}(4)$ counts the following collections of cycles in D_P (with a $-$ sign) which correspond to the 4-cycles in S_4^P :



On the other hand, $f_{D_P}(2, 2)$ counts the following collections of cycles in D_P (with a $+$ sign) which correspond to the permutations in S_4^P which are products of 2 transpositions:



Thus $a_4 = -2 + 2 = 0$.

Returning to our general situation, substituting from (1) we have

$$\begin{aligned}\chi(-2) &= \sum_{i=0}^n a_i 2^{n-i} \\ &= 2^n + \sum_{i=1}^n \left(\sum_{\sum i_j=i} (-1)^{i+r} f_{D_P}(i_1, \dots, i_r) \right) 2^{n-i}.\end{aligned}$$

For each collection of r disjoint directed cycles there are $n-i$ vertices not contained in the cycles. Let $s = n - i + r$ be the total number of connected components of the union of the cycles and the disconnected vertices. We reindex the last sum as a sum over s :

$$\begin{aligned}\chi(-2) &= \sum_{s=1}^n (-1)^{n-s} \left(\sum_{\substack{r, (i_j) \\ s=r+n-\sum i_j}} f_{D_P}(i_1, \dots, i_r) 2^{s-r} \right) \\ &= \sum_{s=1}^n (-1)^{n-s} c_s.\end{aligned}$$

(The sign comes from the fact that $i + r = s - n + 2i$, so $(-1)^{i+r} = (-1)^{n-s}$.) The coefficient

$$c_s = \sum_{\substack{r, (i_j) \\ s=r+n-\sum i_j}} f_{D_P}(i_1, \dots, i_r) 2^{s-r}$$

can also be interpreted as the integer

$$\sum_{\substack{r, (i_j) \\ s=r+n-\sum i_j}} f_{\overline{D_P}}(i_1, \dots, i_r, 1, \dots, 1),$$

(where there are $s-r$ ones) counting the number of collections of disjoint directed cycles of lengths $i_1, \dots, i_r, 1, \dots, 1$ containing *all* the vertices in a directed graph $\overline{D_P}$ obtained from D_P by adding two loops at each vertex (corresponding to the diagonal entries 2 in the matrix $\mathfrak{Z}_P = Z_P + Z_P^t$). Hence c_s is the *number of collections of disjoint directed cycles in the graph $\overline{D_P}$ having precisely s connected components and containing all the vertices*. Note that $s-r$ is the number of loops contained in each such collection.

To summarize, we have proven the following theorem.

Theorem 2. *With the above notation, for all finite partially ordered sets P ,*

$$\det(\mathfrak{Z}_P) = \sum_{s=1}^n (-1)^{n-s} c_s.$$

In terms of permutations in S_n^P , if $1 \leq s \leq n-1$, we have

$$(3) \quad c_s = \sum_{\ell=0}^{s-1} \left| \left\{ \sigma \in S_n^P : \begin{array}{l} \sigma \text{ is a product of } s-\ell \text{ disjoint} \\ \text{cycles of length } > 1 \end{array} \right\} \right| 2^\ell$$

and $c_n = 2^n$.

§3. THE CHAIN CASE

Consider the special case in which P is the chain of length n . Then $P = [n] = \{1, 2, \dots, n\}$ with the usual ordering. All elements of P are comparable, and we have

$$Z_{[n]} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \quad \text{and} \quad M_{[n]} = Z_{[n]}^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ & 1 & -1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix},$$

where all lower-triangular entries in each matrix are zero.

Lemma 2. *With the above notation*

$$\det(\mathfrak{Z}_{[n]}) = \det(\mathfrak{M}_{[n]}) = n + 1.$$

PROOF. The first equality holds by Theorem 1. We prove the second equality by induction on n . We have $\mathfrak{M}_{[1]} = (2)$, hence the lemma holds when $n = 1$. Suppose the lemma holds for all $k < n$, and consider

$$\mathfrak{M}_{[n]} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix},$$

where all remaining entries are zero. Expanding along the first row, we have $\det(\mathfrak{M}_{[n]}) = 2 \det(\mathfrak{M}_{[n-1]}) - (-1) \det(B_n)$ where

$$B_n = \begin{pmatrix} -1 & -1 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & \mathfrak{M}_{[n-2]} & & \\ 0 & & & & \end{pmatrix}.$$

Now expanding along the first column in B_n , we have $\det(B_n) = -\det(\mathfrak{M}_{[n-2]})$. Thus we have

$$\det(\mathfrak{M}_{[n]}) = 2 \det(\mathfrak{M}_{[n-1]}) - \det(\mathfrak{M}_{[n-2]}) = 2n - (n-1) = n+1$$

by applying the inductive hypothesis. \square

All elements of P are comparable in the chain case, and so the digraph $\overline{D_P}$ is complete with double edges between all vertices and double loops at all vertices. Considering permutations written as a product of disjoint cycles, denote by $\Gamma(n; \gamma_1, \gamma_2, \dots, \gamma_n)$ the number of permutations of $[n]$ with exactly γ_i cycles of length i for each $1 \leq i \leq n$. By [S1], Proposition 1.3.2 we have

$$\Gamma(n; \gamma_1, \gamma_2, \dots, \gamma_n) = \frac{n!}{1^{\gamma_1} \gamma_1! 2^{\gamma_2} \gamma_2! \dots n^{\gamma_n} \gamma_n!}.$$

Then, in the chain case, the formula for c_i in (3) becomes

$$c_i = \sum_{\substack{(\gamma_1, \dots, \gamma_n) \\ \sum j \gamma_j = n \\ \sum \gamma_j = i}} \Gamma(n; \gamma_1, \gamma_2, \dots, \gamma_n) 2^{\gamma_1}.$$

The c_i 's may be interpreted as *modified* Stirling numbers of the second kind. Recall that the Stirling numbers of the second kind $\begin{bmatrix} n \\ m \end{bmatrix}$ count the number of permutations of n elements with m disjoint cycles and are given by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{\substack{\sum m_i = m \\ \sum i m_i = n}} \frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \dots k^{m_k} m_k!} = \sum_{\substack{\sum m_i = m \\ \sum i m_i = n}} \Gamma(n; m_1, \dots, m_k).$$

Corollary 1. *The alternating sum of the modified Stirling numbers satisfies the following.*

$$\sum_{i=1}^n (-1)^{n-i} \sum_{\substack{\sum j \gamma_j = n \\ \sum \gamma_j = i}} \Gamma(n; \gamma_1, \dots, \gamma_n) 2^{\gamma_1} = n+1$$

§4. THE BOOLEAN ALGEBRA CASE

Let $[n] = \{1, 2, \dots, n\}$ and consider the poset $P_n = 2^{[n]}$ of subsets of $[n]$ under the inclusion relation. Let \mathfrak{Z}_n be the matrix defined in §1. The main result of this section is the following theorem.

Theorem 3. *If $n \geq 3$ is odd, then $\det(\mathfrak{Z}_n) = 0$. If n is even, then $\det(\mathfrak{Z}_n) = 2^{\alpha_n}$ where $\alpha_2 = 2$, and $\alpha_n = 4\alpha_{n-2} - 2$ for $n \geq 4$ and even.*

PROOF. The proof will rely on several lemmas. First, we identify a particularly useful labelling of $P_n = 2^{[n]}$ for our purposes, and we consider only this labelling in the following. Each subset $A \in P_n$ will be encoded as a binary vector $v(A)$ of length n :

$$v(A) = (v_n, v_{n-1}, \dots, v_1)$$

where

$$v_i = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if not.} \end{cases}$$

Our labelling of P_n induces the usual numerical ordering when we interpret $v(A)$ as the binary expansion of an integer m , with $0 \leq m \leq 2^n - 1$.

Using this labelling yields an interesting recursive structure in the matrices \mathfrak{Z}_n .

Lemma 3. *The matrices Z_n and \mathfrak{Z}_n have the following properties.*

- (1) *The entries of Z_n above the diagonal are the first 2^n rows in the Pascal triangle modulo 2.*
- (2) *For $n \geq 2$, Z_n and \mathfrak{Z}_n have block decompositions:*

$$Z_n = \begin{pmatrix} Z_{n-1} & Z_{n-1} \\ 0 & Z_{n-1} \end{pmatrix} \quad \text{and} \quad \mathfrak{Z}_n = \begin{pmatrix} \mathfrak{Z}_{n-1} & Z_{n-1} \\ Z_{n-1}^t & \mathfrak{Z}_{n-1} \end{pmatrix}.$$

(This statement also holds with $n = 1$ if we take $\mathfrak{Z}_0 = 2$, $Z_0 = 1$.)

- (3) *The Z_n matrix sequence can be generated by a recursive procedure as follows.*

Given Z_{n-1} , to form Z_n we replace each entry 1 by a 2×2 block $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and each entry 0 by a 2×2 zero matrix. From part (1), we get a similar recursive procedure for the \mathfrak{Z}_n sequence.

PROOF. These properties follow directly from the definition of the \mathfrak{Z}_n matrices and the properties of the preferred ordering on P_n . \square

To evaluate the determinant $\det(\mathfrak{Z}_n)$, we follow the general advice of [K] and introduce parameters in the matrix entries. Specifically, we consider the matrix:

$$\mathfrak{Z}_n(x, y) = xZ_n + yZ_n^t = \begin{pmatrix} xZ_{n-1} + yZ_{n-1}^t & xZ_{n-1} \\ yZ_{n-1}^t & xZ_{n-1} + yZ_{n-1}^t \end{pmatrix}$$

Using the Maple computer algebra system, the determinants of the first few of these matrices are found to be:

$$\begin{array}{rcl}
 n & & \det(\mathfrak{Z}_n(x, y)) \\
 (4) & & \\
 & 1 & x^2 + xy + y^2 \\
 & 2 & (x + y)^2(x^2 - xy + y^2) \\
 & 3 & (x - y)^2(x^2 + xy + y^2)^3 \\
 & 4 & (x + y)^6(x^2 - xy + y^2)^5 \\
 & 5 & (x - y)^{10}(x^2 + xy + y^2)^{11}
 \end{array}$$

An interesting recurrence explains the patterns evident in these examples.

Lemma 4. *The determinants of the $\mathfrak{Z}_n(x, y)$ matrices are related by the following recurrence:*

$$\det(\mathfrak{Z}_{n+2}(x, y)) = (\det(\mathfrak{Z}_n(x, y)))^2 \det(\mathfrak{Z}_{n+1}(x, -y)).$$

(Note that the right side involves both \mathfrak{Z}_{n+1} and \mathfrak{Z}_n and that y is negated in the second factor.)

PROOF. We use part (2) of Lemma 3 twice to form the following block decomposition of $\mathfrak{Z}_{n+2}(x, y)$:

$$\mathfrak{Z}_{n+2}(x, y) = \begin{pmatrix} xZ_n + yZ_n^t & xZ_n & xZ_n & xZ_n \\ yZ_n^t & xZ_n + yZ_n^t & 0 & xZ_n \\ yZ_n^t & 0 & xZ_n + yZ_n^t & xZ_n \\ yZ_n^t & yZ_n^t & yZ_n^t & xZ_n + yZ_n^t \end{pmatrix}$$

(where each entry is a block of size $2^n \times 2^n$). To evaluate the determinant, we perform block-wise row and column operations. To simplify the notation, we write $Z = Z_n$. First subtract row 4 from each of the first three rows to obtain:

$$\begin{pmatrix} xZ & xZ - yZ^t & xZ - yZ^t & -yZ^t \\ 0 & xZ & -yZ^t & -yZ^t \\ 0 & -yZ^t & xZ & -yZ^t \\ yZ^t & yZ^t & yZ^t & xZ + yZ^t \end{pmatrix}.$$

Subtract column 1 from each of the columns 2,3,4 to obtain:

$$\begin{pmatrix} xZ & -yZ^t & -yZ^t & -xZ - yZ^t \\ 0 & xZ & -yZ^t & -yZ^t \\ 0 & -yZ^t & xZ & -yZ^t \\ yZ^t & 0 & 0 & xZ \end{pmatrix}.$$

Next, subtract column 2 from column 4:

$$\begin{pmatrix} xZ & -yZ^t & -yZ^t & -xZ \\ 0 & xZ & -yZ^t & -xZ - yZ^t \\ 0 & -yZ^t & xZ & 0 \\ yZ^t & 0 & 0 & xZ \end{pmatrix}.$$

Add column 1 to column 4, then add row 4 in the resulting matrix to row 2:

$$\mathfrak{Z}' = \begin{pmatrix} xZ & -yZ^t & -yZ^t & 0 \\ yZ^t & xZ & -yZ^t & 0 \\ 0 & -yZ^t & xZ & 0 \\ yZ^t & 0 & 0 & xZ + yZ^t \end{pmatrix}.$$

Expanding along the last column we obtain

$$(5) \quad \det(\mathfrak{Z}_{n+2}(x, y)) = \det(\mathfrak{Z}') = \det(xZ + yZ^t) \det \begin{pmatrix} xZ & -yZ^t & -yZ^t \\ yZ^t & xZ & -yZ^t \\ 0 & -yZ^t & xZ \end{pmatrix}.$$

The first factor on the right of (5) is $\det(\mathfrak{Z}_n(x, y))$. Continuing with the 3×3 matrix, subtract column 3 from column 2:

$$\begin{pmatrix} xZ & 0 & -yZ^t \\ yZ^t & xZ + yZ^t & -yZ^t \\ 0 & -xZ - yZ^t & xZ \end{pmatrix},$$

then add row three to row 2:

$$\begin{pmatrix} xZ & 0 & -yZ^t \\ yZ^t & 0 & xZ - yZ^t \\ 0 & -xZ - yZ^t & xZ \end{pmatrix}.$$

Expanding along column 2, we have

$$\begin{aligned} \det(\mathfrak{Z}_{n+2}(x, y)) &= \det(\mathfrak{Z}_n(x, y))^2 \det \begin{pmatrix} xZ & -yZ^t \\ yZ^t & xZ - yZ^t \end{pmatrix} \\ &= \det(\mathfrak{Z}_n(x, y))^2 \det \begin{pmatrix} xZ - yZ^t & -yZ^t \\ xZ & xZ - yZ^t \end{pmatrix} \\ &= \det(\mathfrak{Z}_n(x, y))^2 \det(\mathfrak{Z}_{n+1}(x, -y)), \end{aligned}$$

as claimed. For the last equality, we perform row and column interchanges to put the final matrix shown into the form:

$$\begin{pmatrix} xZ - yZ^t & xZ \\ -yZ^t & xZ - yZ^t \end{pmatrix}$$

required for $\mathfrak{Z}_{n+1}(x, -y)$. \square

From the initial cases computed with Maple in (4) and the recurrence from Lemma 4, we see that there are nonnegative integers α_n, β_n such that

$$(6) \quad \det(\mathfrak{Z}_n(x, y)) = (x + (-1)^n y)^{\alpha_n} (x^2 - (-1)^n xy + y^2)^{\beta_n}.$$

Moreover, the recurrence from Lemma 4 implies that

$$(7) \quad \begin{cases} \alpha_{n+2} = 2\alpha_n + \alpha_{n+1}, \\ \beta_{n+2} = 2\beta_n + \beta_{n+1}. \end{cases}$$

We also have the following fact that is evident from (4):

Lemma 5. *For all $n \geq 1$, $\alpha_n = \beta_n + (-1)^{n+1}$.*

PROOF. This follows by induction. The base cases come from the Maple computations in (4) above: $\alpha_1 = 0, \beta_1 = 1$, and $\alpha_2 = 2, \beta_2 = 1$. For the induction step, assume that the claim of the lemma has been proved for all $\ell \leq k+1$. Then subtracting the two recurrences from (7) shows that

$$\alpha_{k+2} - \beta_{k+2} = 2(\alpha_k - \beta_k) + \alpha_{k+1} - \beta_{k+1} = 2(-1)^{k+1} + (-1)^{k+2} = (-1)^{k+1} = (-1)^{k+3}. \quad \square$$

We are now ready to conclude the proof of our main theorem. To determine the determinant of the original $\mathfrak{Z}_n = \mathfrak{Z}_n(1, 1)$, we simply substitute $x = y = 1$ in (6). The factors of $x - y$ show immediately that $\det(\mathfrak{Z}_n) = 0$ if n is odd. Moreover, when n is even we have $\det(\mathfrak{Z}_n) = 2^{\alpha_n}$. We solve the first recurrence in (7) for α_n by the standard method for second order linear recurrences with constant coefficients. The characteristic equation is $r^2 - r - 2 = 0$, whose roots are $r = 2, -1$. Hence $\alpha_n = c_1(2)^n + c_2(-1)^n$ for some constants c_1, c_2 . The initial conditions $\alpha_1 = 0, \alpha_2 = 2$ show that $c_1 = 1/3, c_2 = 2/3$. Hence:

$$\alpha_n = \frac{1}{3}(2)^n + \frac{2}{3}(-1)^n.$$

Hence if n , and therefore also $n+2$, are even, we have

$$\alpha_{n+2} = \frac{1}{3}(2)^{n+2} + \frac{2}{3} = 4 \left(\frac{1}{3}(2)^n + \frac{2}{3} \right) - 2 = 4\alpha_n - 2. \quad \square$$

Corollary 2. *If n is even, then*

$$\det(\mathfrak{Z}_n) = 2^{\frac{2^n+2}{3}}.$$

REFERENCES

- [H] F. Harary, *The Determinant of the Adjacency Matrix of a Graph*, SIAM Review **4**, no. 3 (1962), 202-210.
- [K] C. Krattenthaler, *Advanced Determinant Calculus*, Sm. Lothar. Combin. **42** (1999).
- [M] A. Mowshowitz, *The Characteristic Polynomial of a Graph*, J. Comb Th. Ser. B **12** (1972), 177-193.
- [S1] R. Stanley, *Enumerative Combinatorics, vol. 1*, Cambridge University Press, Cambridge, 1997.
- [S2] R. Stanley, *Enumerative Combinatorics, vol. 2*, Cambridge University Press, Cambridge, 1999.
- [Sa] B.E. Sagan, *Why the Characteristic Polynomial Factors*, Bull. AMS **36**, no. 2 (1999), 113-133.

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